

Contents

Preface to the Second German Edition	vii
Preface to the First German Edition	vii
Acknowledgments	x
Advice to the reader	x
A Infinite Products and Partial Fraction Series 1	
1 Infinite Products of Holomorphic Functions	3
§1. Infinite Products	4
1. Infinite products of numbers	4
2. Infinite products of functions	6
§2. Normal Convergence	7
1. Normal convergence	7
2. Normally convergent products of holomorphic functions	9
3. Logarithmic differentiation	10
§3. The Sine Product $\sin \pi z = \pi z \prod_{\nu=1}^{\infty} (1 - z^2/\nu^2)$	12
1. Standard proof	12
2. Characterization of the sine by the duplication formula	14
3. Proof of Euler's formula using Lemma 2	15
4*. Proof of the duplication formula for Euler's product, following Eisenstein	16
5. On the history of the sine product	17

§4*. Euler Partition Products	18
1. Partitions of natural numbers and Euler products .	19
2. Pentagonal number theorem. Recursion formulas for $p(n)$ and $\sigma(n)$	20
3. Series expansion of $\prod_{\nu=1}^{\infty}(1+q^{\nu}z)$ in powers of z .	22
4. On the history of partitions and the pentagonal number theorem	24
§5*. Jacobi's Product Representation of the Series $J(z, q) := \sum_{\nu=-\infty}^{\infty} q^{\nu^2} z^{\nu}$	25
1. Jacobi's theorem	25
2. Discussion of Jacobi's theorem	26
3. On the history of Jacobi's identity	28
Bibliography	30
 2 The Gamma Function	33
§1. The Weierstrass Function $\Delta(z) = ze^{\gamma z} \prod_{\nu \geq 1} (1+z/\nu)e^{-z/\nu}$	36
1. The auxiliary function $H(z) := z \prod_{\nu=1}^{\infty} (1+z/\nu)e^{-z/\nu}$	36
2. The entire function $\Delta(z) := e^{\gamma z} H(z)$	37
§2. The Gamma Function	39
1. Properties of the Γ -function	39
2. Historical notes	41
3. The logarithmic derivative	42
4. The uniqueness problem	43
5. Multiplication formulas	45
6*. Hölder's theorem	46
7*. The logarithm of the Γ -function	47
§3. Euler's and Hankel's Integral Representations of $\Gamma(z)$	49
1. Convergence of Euler's integral	49
2. Euler's theorem	51
3*. The equation	52
4*. Hankel's loop integral	53
§4. Stirling's Formula and Gudermann's Series	55
1. Stieltjes's definition of the function $\mu(z)$	56
2. Stirling's formula	58
3. Growth of $ \Gamma(x+iy) $ for $ y \rightarrow \infty$	59
4*. Gudermann's series	60
5*. Stirling's series	61
6*. Delicate estimates for the remainder term	63
7*. Binet's integral	64
8*. Lindelöf's estimate	66
§5. The Beta Function	67
1. Proof of Euler's identity	68
2. Classical proofs of Euler's identity	69
Bibliography	70

3	Entire Functions with Prescribed Zeros	73
§1.	The Weierstrass Product Theorem for \mathbb{C}	74
1.	Divisors and principal divisors	74
2.	Weierstrass products	75
3.	Weierstrass factors	76
4.	The Weierstrass product theorem	77
5.	Consequences	78
6.	On the history of the product theorem	79
§2.	Discussion of the Product Theorem	80
1.	Canonical products	81
2.	Three classical canonical products	82
3.	The σ -function	83
4.	The φ -function	85
5*.	An observation of Hurwitz	85
	Bibliography	86
4*	Holomorphic Functions with Prescribed Zeros	89
§1.	The Product Theorem for Arbitrary Regions	89
1.	Convergence lemma	90
2.	The product theorem for special divisors	91
3.	The general product theorem	92
4.	Second proof of the general product theorem	92
5.	Consequences	93
§2.	Applications and Examples	94
1.	Divisibility in the ring $\mathcal{O}(G)$. Greatest common divisors	94
2.	Examples of Weierstrass products	96
3.	On the history of the general product theorem	97
4.	Glimpses of several variables	97
§3.	Bounded Functions on \mathbb{E} and Their Divisors	99
1.	Generalization of Schwarz's lemma	99
2.	Necessity of the Blaschke condition	100
3.	Blaschke products	100
4.	Bounded functions on the right half-plane	102
	Appendix to Section 3: Jensen's Formula	102
	Bibliography	104
5	Iss'sa's Theorem. Domains of Holomorphy	107
§1.	Iss'sa's Theorem	107
1.	Bers's theorem	108
2.	Iss'sa's theorem	109
3.	Proof of the lemma	109
4.	Historical remarks on the theorems of Bers and Iss'sa	110
5*.	Determination of all the valuations on $\mathcal{M}(G)$	111

§2.	Domains of Holomorphy	112
1.	A construction of Goursat	113
2.	Well-distributed boundary sets. First proof of the existence theorem	115
3.	Discussion of the concept of domains of holomorphy	116
4.	Peripheral sets. Second proof of the existence theorem	118
5.	On the history of the concept of domains of holomorphy	119
6.	Glimpse of several variables	120
§3.	Simple Examples of Domains of Holomorphy	120
1.	Examples for \mathbb{E}	120
2.	Lifting theorem	122
3.	Cassini regions and domains of holomorphy	122
	Bibliography	123
 6	Functions with Prescribed Principal Parts	125
§1.	Mittag-Leffler's Theorem for \mathbb{C}	126
1.	Principal part distributions	126
2.	Mittag-Leffler series	127
3.	Mittag-Leffler's theorem.	128
4.	Consequences	128
5.	Canonical Mittag-Leffler series. Examples	129
6.	On the history of Mittag-Leffler's theorem for \mathbb{C} . .	130
§2.	Mittag-Leffler's Theorem for Arbitrary Regions	131
1.	Special principal part distributions	131
2.	Mittag-Leffler's general theorem	132
3.	Consequences	133
4.	On the history of Mittag-Leffler's general theorem .	134
5.	Glimpses of several variables	135
§3*.	Ideal Theory in Rings of Holomorphic Functions	135
1.	Ideals in $\mathcal{O}(G)$ that are not finitely generated . . .	136
2.	Wedderburn's lemma (representation of 1)	136
3.	Linear representation of the gcd. Principal ideal theorem	138
4.	Nonvanishing ideals	138
5.	Main theorem of the ideal theory of $\mathcal{O}(G)$	139
6.	On the history of the ideal theory of holomorphic functions	140
7.	Glimpses of several variables	141
	Bibliography	142

B Mapping Theory	145
7 The Theorems of Montel and Vitali	147
§1. Montel's Theorem	148
1. Montel's theorem for sequences	148
2. Proof of Montel's theorem	150
3. Montel's convergence criterion	150
4. Vitali's theorem	150
5*. Pointwise convergent sequences of holomorphic functions	151
§2. Normal Families	152
1. Montel's theorem for normal families	152
2. Discussion of Montel's theorem	153
3. On the history of Montel's theorem	154
4*. Square-integrable functions and normal families	154
§3*. Vitali's Theorem	156
1. Convergence lemma	156
2. Vitali's theorem (final version)	157
3. On the history of Vitali's theorem	158
§4*. Applications of Vitali's theorem	159
1. Interchanging integration and differentiation	159
2. Compact convergence of the Γ -integral	160
3. Müntz's theorem	161
§5. Consequences of a Theorem of Hurwitz	162
Bibliography	164
8 The Riemann Mapping Theorem	167
§1. Integral Theorems for Homotopic Paths	168
1. Fixed-endpoint homotopic paths	168
2. Freely homotopic closed paths	169
3. Null homotopy and null homology	170
4. Simply connected domains	171
5*. Reduction of the integral theorem 1 to a lemma	172
6*. Proof of Lemma 5*	174
§2. The Riemann Mapping Theorem	175
1. Reduction to Q -domains	175
2. Existence of holomorphic injections	177
3. Existence of expansions	177
4. Existence proof by means of an extremal principle	178
5. On the uniqueness of the mapping function	179
6. Equivalence theorem	180
§3. On the History of the Riemann Mapping Theorem	181
1. Riemann's dissertation	181
2. Early history	183
3. From Carathéodory-Koebe to Fejér-Riesz	184

4.	Carathéodory's final proof	184
5.	Historical remarks on uniqueness and boundary behavior	186
6.	Glimpses of several variables	187
§4.	Isotropy Groups of Simply Connected Domains	188
1.	Examples	188
2.	The group $\text{Aut}_a G$ for simply connected domains $G \neq \mathbb{C}$	189
3*.	Mapping radius. Monotonicity theorem	189
Appendix to Chapter 8: Carathéodory-Koebe Theory		191
§1.	Simple Properties of Expansions	191
1.	Expansion lemma	191
2.	Admissible expansions. The square root method . .	192
3*.	The crescent expansion	193
§2.	The Carathéodory-Koebe Algorithm	194
1.	Properties of expansion sequences	194
2.	Convergence theorem	195
3.	Koebe families and Koebe sequences	196
4.	Summary. Quality of convergence	197
5.	Historical remarks. The competition between Carathéodory and Koebe	197
§3.	The Koebe Families \mathcal{K}_m and \mathcal{K}_∞	198
1.	A lemma	198
2.	The families \mathcal{K}_m and \mathcal{K}_∞	199
Bibliography for Chapter 8 and the Appendix		201
9	Automorphisms and Finite Inner Maps	203
§1.	Inner Maps and Automorphisms	203
1.	Convergent sequences in $\text{Hol } G$ and $\text{Aut } G$	204
2.	Convergence theorem for sequences of automorphisms	204
3.	Bounded homogeneous domains	205
4*.	Inner maps of \mathbb{H} and homotheties	206
§2.	Iteration of Inner Maps	206
1.	Elementary properties	207
2.	H. Cartan's theorem	207
3.	The group $\text{Aut}_a G$ for bounded domains	208
4.	The closed subgroups of the circle group	209
5*.	Automorphisms of domains with holes. Annulus theorem	210
§3.	Finite Holomorphic Maps	211
1.	Three general properties	212
2.	Finite inner maps of \mathbb{E}	212
3.	Boundary lemma for annuli	213

4.	Finite inner maps of annuli	215
5.	Determination of all the finite maps between annuli	216
§4*.	Radó's Theorem. Mapping Degree	217
1.	Closed maps. Equivalence theorem	217
2.	Winding maps	218
3.	Radó's theorem	219
4.	Mapping degree	220
5.	Glimpses	221
	Bibliography	221
C	Selecta	223
10	The Theorems of Bloch, Picard, and Schottky	225
§1.	Bloch's Theorem	226
1.	Preparation for the proof	226
2.	Proof of Bloch's theorem	227
3*.	Improvement of the bound by the solution of an extremal problem	228
4*.	Ahlfors's theorem	230
5*.	Landau's universal constant	232
§2.	Picard's Little Theorem	233
1.	Representation of functions that omit two values .	233
2.	Proof of Picard's little theorem	234
3.	Two amusing applications	235
§3.	Schottky's Theorem and Consequences	236
1.	Proof of Schottky's theorem	237
2.	Landau's sharpened form of Picard's little theorem	238
3.	Sharpened forms of Montel's and Vitali's theorems	239
§4.	Picard's Great Theorem	240
1.	Proof of Picard's great theorem	240
2.	On the history of the theorems of this chapter .	240
	Bibliography	241
11	Boundary Behavior of Power Series	243
§1.	Convergence on the Boundary	244
1.	Theorems of Fatou, M. Riesz, and Ostrowski . .	244
2.	A lemma of M. Riesz	245
3.	Proof of the theorems in 1	247
4.	A criterion for nonextendibility	248
	Bibliography for Section 1	249
§2.	Theory of Overconvergence. Gap Theorem	249

1.	Overconvergent power series	249
2.	Ostrowski's overconvergence theorem	250
3.	Hadamard's gap theorem	252
4.	Porter's construction of overconvergent series	253
5.	On the history of the gap theorem	254
6.	On the history of overconvergence	255
7.	Glimpses	255
	Bibliography for Section 2	256
§3.	A Theorem of Fatou-Hurwitz-Pólya	257
1.	Hurwitz's proof	258
2.	Glimpses	259
	Bibliography for Section 3	259
§4.	An Extension Theorem of Szegő	260
1.	Preliminaries for the proof of (Sz)	260
2.	A lemma	262
3.	Proof of (Sz)	263
4.	An application	263
5.	Glimpses	265
	Bibliography for Section 4	266
12	Runge Theory for Compact Sets	267
§1.	Techniques	268
1.	Cauchy integral formula for compact sets	269
2.	Approximation by rational functions	271
3.	Pole-shifting theorem	272
§2.	Runge Theory for Compact Sets	273
1.	Runge's approximation theorem	273
2.	Consequences of Runge's little theorem	275
3.	Main theorem of Runge theory for compact sets	276
§3.	Applications of Runge's Little Theorem	278
1.	Pointwise convergent sequences of polynomials that do not converge compactly everywhere	278
2.	Holomorphic imbedding of the unit disc in \mathbb{C}^3	281
§4.	Discussion of the Cauchy Integral Formula for Compact Sets	283
1.	Final form of Theorem 1.1	284
2.	Circuit theorem	285
	Bibliography	287
13	Runge Theory for Regions	289
§1.	Runge's Theorem for Regions	290
1.	Filling in compact sets. Runge's proof of Mittag-Leffler's theorem	291
2.	Runge's approximation theorem	292
3.	Main theorem of Cauchy function theory	292

4.	On the theory of holes	293
5.	On the history of Runge theory	294
§2.	Runge Pairs	295
1.	Topological characterization of Runge pairs	295
2.	Runge hulls	296
3.	Homological characterization of Runge hulls. The Behnke-Stein theorem	297
4.	Runge regions	298
5*.	Approximation and holomorphic extendibility	299
§3.	Holomorphically Convex Hulls and Runge Pairs	300
1.	Properties of the hull operator	300
2.	Characterization of Runge pairs by means of holomorphically convex hulls	302
Appendix: On the Components of Locally Compact Spaces. Šura-Bura's Theorem		303
1.	Components	303
2.	Existence of open compact sets	304
3.	Filling in holes	305
4.	Proof of Šura-Bura's theorem	305
Bibliography		306
14	Invariance of the Number of Holes	309
§1.	Homology Theory. Separation Lemma	309
1.	Homology groups. The Betti number	310
2.	Induced homomorphisms. Natural properties	311
3.	Separation of holes by closed paths	312
§2.	Invariance of the Number of Holes. Product Theorem for Units	313
1.	On the structure of the homology groups	313
2.	The number of holes and the Betti number	314
3.	Normal forms of multiply connected domains (report)	315
4.	On the structure of the multiplicative group $\mathcal{O}(G)^\times$	316
5.	Glimpses	318
Bibliography		318
Correction to: Classical Topics in Complex Function Theory		E1
Short Biographies		321
Symbol Index		329
Name Index		331
Subject Index		337