

Contents

Introduction	1
------------------------	---

CHAPTER ONE

Grassmann algebra

1.1. Tensor products	8
1.2. Graded algebras	11
1.3. The exterior algebra of a vectorspace	13
1.4. Alternating forms and duality	16
1.5. Interior multiplications	21
1.6. Simple m -vectors	23
1.7. Inner products	27
1.8. Mass and comass	38
1.9. The symmetric algebra of a vectorspace	41
1.10. Symmetric forms and polynomial functions	43

CHAPTER TWO

General measure theory

2.1. Measures and measurable sets	51
2.1.1. Numerical summation	51
2.1.2. – 3. Measurable sets	53
2.1.4. – 5. Measure hulls	56
2.1.6. Ulam numbers	58
2.2. Borel and Suslin sets	59
2.2.1. Borel families	59
2.2.2. – 3. Approximation by closed subsets	60
2.2.4. Nonmeasurable sets	62
2.2.5. Radon measures	62
2.2.6. The space of sequences of positive integers	63
2.2.7. – 9. Lipschitzian maps	63
2.2.10. – 13. Suslin sets	65
2.2.14. – 15. Borel and Baire functions	70
2.2.16. Separability of supports	71
2.2.17. Images of Radon measures	72
2.3. Measurable functions	72
2.3.1. – 2. Basic properties	72
2.3.3. – 7. Approximation theorems	76
2.3.8. – 10. Spaces of measurable functions	78

2.4.	Lebesgue integration	80
2.4.1. – 5.	Basic properties	80
2.4.6. – 9.	Limit theorems	84
2.4.10. – 11.	Integrals over subsets	85
2.4.12. – 17.	Lebesgue spaces	86
2.4.18.	Compositions and image measures	90
2.4.19.	Jensen's inequality	91
2.5.	Linear functionals	91
2.5.1.	Lattices of functions	91
2.5.2. – 6.	Daniell integrals	92
2.5.7. – 12.	Linear functionals on Lebesgue spaces	98
2.5.13. – 15.	Riesz's representation theorem	106
2.5.16.	Curve length	109
2.5.17. – 18.	Riemann-Stieltjes integration	110
2.5.19.	Spaces of Daniell integrals	113
2.5.20.	Decomposition of Daniell integrals	114
2.6.	Product measures	114
2.6.1. – 4.	Fubini's theorem	114
2.6.5.	Lebesgue measure	119
2.6.6.	Infinite cartesian products	120
2.6.7.	Integration by parts	121
2.7.	Invariant measures	121
2.7.1. – 3.	Definitions	121
2.7.4. – 13.	Existence and uniqueness of invariant integrals	123
2.7.14. – 15.	Covariant measures are Radon measures	131
2.7.16.	Examples	133
2.7.17.	Nonmeasurable sets	141
2.7.18.	L_1 continuity of group actions	141
2.8.	Covering theorems	141
2.8.1. – 3.	Adequate families	141
2.8.4. – 8.	Coverings with enlargement	143
2.8.9. – 15.	Centered ball coverings	145
2.8.16. – 20.	Vitali relations	151
2.9.	Derivates	152
2.9.1. – 5.	Existence of derivates	152
2.9.6. – 10.	Indefinite integrals	155
2.9.11. – 13.	Density and approximate continuity	158
2.9.14. – 18.	Additional results on derivation using centered balls	159
2.9.19. – 25.	Derivatives of curves with finite length	163
2.10.	Carathéodory's construction	169
2.10.1.	The general construction	169
2.10.2. – 6.	The measures $\mathcal{H}^m, \mathcal{J}^m, \mathcal{T}^m, \mathcal{G}^m, \mathcal{C}^m, \mathcal{J}_t^m, \mathcal{Q}_t^m$	171
2.10.7.	Relation to Riemann-Stieltjes integration	174
2.10.8. – 11.	Partitions and multiplicity integrals	175
2.10.12. – 14.	Curve length	176
2.10.15. – 16.	Integralgeometric measures	178
2.10.17. – 19.	Densities	179
2.10.20.	Remarks on approximating measures	181

2.10.21.	Spaces of Lipschitzian functions and closed subsets	182
2.10.22. – 23.	Approximating measures of increasing sequences	184
2.10.24.	Direct construction of the upper integral	186
2.10.25. – 27.	Integrals of measures of counterimages	188
2.10.28. – 29.	Sets of Cantor type	191
2.10.30. – 31.	Steiner symmetrization	195
2.10.32. – 42.	Inequalities between basic measures	196
2.10.43. – 44.	Lipschitzian extension of functions	201
2.10.45. – 46.	Cartesian products	202
2.10.47. – 48.	Subsets of finite Hausdorff measure	204

CHAPTER THREE

Rectifiability

3.1.	Differentials and tangents	209
3.1.1. – 10.	Differentiation and approximate differentiation	209
3.1.11.	Higher differentials	218
3.1.12. – 13.	Partitions of unity	223
3.1.14. – 17.	Differentiable extension of functions	225
3.1.18.	Factorization of maps near generic points	229
3.1.19. – 20.	Submanifolds of Euclidean space	231
3.1.21.	Tangent vectors	233
3.1.22.	Relative differentiation	235
3.1.23.	Local flattening of a submanifold	236
3.1.24.	Analytic functions	237
3.2.	Area and coarea of Lipschitzian maps	241
3.2.1.	Jacobians	241
3.2.2. – 7.	Area of maps of Euclidean spaces	242
3.2.8. – 12.	Coarea of maps of Euclidean spaces	247
3.2.13.	Applications; Euler's function Γ	250
3.2.14. – 15.	Rectifiable sets	251
3.2.16. – 19.	Approximate tangent vectors and differentials	252
3.2.20. – 22.	Area and coarea of maps of rectifiable sets	256
3.2.23. – 24.	Cartesian products	260
3.2.25. – 26.	Equality of measures of rectifiable sets	261
3.2.27.	Areas of projections of rectifiable sets	262
3.2.28.	Examples	263
3.2.29.	Rectifiable sets and manifolds of class 1	267
3.2.30. – 33.	Further results on coarea	268
3.2.34. – 40.	Steiner's formula and Minkowski content	271
3.2.41. – 44.	Brunn-Minkowski theorem	277
3.2.45.	Relations between the measures $\mathcal{Q}_t^m$	279
3.2.46.	Hausdorff measures in Riemannian manifolds	280
3.2.47. – 49.	Integral geometry on spheres	282
3.3.	Structure theory	287
3.3.1. – 4.	Tangential properties of arbitrary Suslin sets	287
3.3.5. – 18.	Rectifiability and projections	292
3.3.19. – 21.	Examples of unrectifiable sets	302
3.3.22.	Rectifiability and density	309

3.4.	Some properties of highly differentiable functions	310
3.4.1. – 4.	Measures of $f \{x: \dim \operatorname{im} Df(x) \leq v\}$	310
3.4.5. – 12.	Analytic varieties	318

CHAPTER FOUR

Homological integration theory

4.1.	Differential forms and currents	343
4.1.1.	Distributions	343
4.1.2. – 4.	Regularization	346
4.1.5.	Distributions representable by integration	349
4.1.6.	Differential forms and m -vectorfields	351
4.1.7.	Currents	355
4.1.8.	Cartesian products	360
4.1.9. – 10.	Homotopies	363
4.1.11.	Joins, oriented simplexes	364
4.1.12. – 19.	Flat chains	367
4.1.20. – 21.	Relation to integral geometry measure	378
4.1.22. – 23.	Polyhedral chains and flat approximation	379
4.1.24. – 28.	Rectifiable currents	380
4.1.29.	Lipschitz neighborhood retracts	386
4.1.30.	Transformation formula	387
4.1.31.	Oriented submanifolds	389
4.1.32.	Projective maps and polyhedral chains	392
4.1.33.	Duality formulae	394
4.1.34.	Lie product of vectorfields	394
4.2.	Deformations and compactness	395
4.2.1.	Slicing normal currents by real valued functions	395
4.2.2.	Maps with singularities	396
4.2.3. – 6.	Cubical subdivisions	398
4.2.7. – 9.	Deformation theorem	404
4.2.10.	Isoperimetric inequality	408
4.2.11. – 14.	Flat chains and integral geometric measure	408
4.2.15. – 16.	Closure theorem	411
4.2.17. – 18.	Compactness theorem	414
4.2.19. – 24.	Approximation by polyhedral chains	415
4.2.25.	Indecomposable integral currents	420
4.2.26.	Flat chains modulo v	423
4.2.27.	Locally rectifiable currents	432
4.2.28. – 29.	Analytic chains	433
4.3.	Slicing	435
4.3.1. – 8.	Slicing flat chains by maps into $\mathbf{R}^n$	435
4.3.9. – 12.	Homotopies, continuity of slices	445
4.3.13.	Slicing by maps into manifolds	451
4.3.14.	Oriented cones	452
4.3.15.	Oriented cylinders	455
4.3.16. – 19.	Oriented tangent cones	455
4.3.20.	Intersections of flat chains	460
4.4.	Homology groups	463
4.4.1.	Homology theory with coefficient group $\mathbf{Z}$	463
4.4.2. – 3.	Isoperimetric inequalities	466

4.4.4.	Compactness properties of homology classes	470
4.4.5. – 6.	Homology theories with coefficient groups $\mathbf{R}$ and $\mathbf{Z}_v$	472
4.4.7.	Two simple examples	473
4.4.8.	Homotopy groups of cycle groups	474
4.4.9.	Cohomology groups	474
4.5.	Normal currents of dimension n in $\mathbf{R}^n$	474
4.5.1. – 4.	Sets with locally finite perimeter	474
4.5.5.	Exterior normals	477
4.5.6.	Gauss-Green theorem	478
4.5.7. – 10.	Functions corresponding to locally normal currents	480
4.5.11. – 12.	Densities and locally finite perimeter	506
4.5.13. – 17.	Examples and applications	509

CHAPTER FIVE

Applications to the calculus of variations

5.1.	Integrands and minimizing currents	515
5.1.1.	Parametric integrands and integrals	515
5.1.2.	Ellipticity of parametric integrands	517
5.1.3.	Convexity, parametric Legendre condition	518
5.1.4.	Diffeomorphic invariance of ellipticity	519
5.1.5.	Lowersemicontinuity of the integral	519
5.1.6.	Minimizing currents	521
5.1.7. – 8.	Isotopic deformations, variations	524
5.1.9.	Nonparametric integrands	527
5.1.10.	Nonparametric Legendre condition	529
5.1.11.	Euler-Lagrange formulae	530
5.2.	Regularity of solutions of certain differential equations	532
5.2.1. – 2.	L_2 and Hölder conditions	532
5.2.3.	Strongly elliptic systems	534
5.2.4.	Sobolev's inequality	537
5.2.5. – 6.	Generalized harmonic functions	538
5.2.7. – 10.	Convolutions with essentially homogeneous functions	541
5.2.11. – 13.	Elementary solutions	547
5.2.14.	Hölder estimate for linear systems	552
5.2.15. – 18.	Nonparametric variational problems	554
5.2.19.	Maxima of real valued solutions	560
5.2.20.	One dimensional variational problems	564
5.3.	Excess and smoothness	565
5.3.1. – 6.	Estimates involving excess	565
5.3.7.	A limiting process	581
5.3.8. – 13.	The decrease of excess	585
5.3.14. – 17.	Regularity of minimizing currents	606
5.3.18. – 19.	Minimizing currents of dimension m in $\mathbf{R}^{m+1}$	613
5.3.20.	Minimizing currents of dimension 1 in $\mathbf{R}^n$	617
5.3.21.	Minimizing flat chains modulo v	619
5.4.	Further results on area minimizing currents	619
5.4.1.	Terminology	619
5.4.2.	Weak convergence of variation measures	620

5.4.3. – 5.	Density ratios and tangent cones	621
5.4.6. – 7.	Regularity of area minimizing currents	628
5.4.8. – 9.	Cartesian products	631
5.4.10. – 14.	Study of cones by differential geometry	632
5.4.15. – 16.	Currents of dimension m in $\mathbf{R}^{m+1}$	644
5.4.17.	Lack of uniqueness and symmetry	647
5.4.18.	Nonparametric surfaces, Bernstein's theorem	649
5.4.19.	Holomorphic varieties	651
5.4.20.	Boundary regularity	654
Bibliography		655
Glossary of some standard notations		669
List of basic notations defined in the text		670
Index		672