

Contents

1	Sequences of Numbers.....	1
1	1 Convergence and Introductory Examples	1
1	1.1 Definition of a Sequence and Trivial (Pre-limit) Properties	1
1	1.2 Convergence of a Sequence	3
1	2 Common Properties of Convergent Sequences	8
1	2.1 Uniqueness of the Limit	8
1	2.2 Comparison Properties	8
1	2.3 Arithmetic and Analytic Properties	11
1	3 Special Properties of Convergent Sequences	13
1	3.1 Convergence of Function and Corresponding Sequence	13
1	3.2 Relationship Between Convergence and Boundedness	13
1	3.3 Subsequences and Their Convergence. Bolzano-Weierstrass Theorem.....	14
1	3.4 Cauchy Criterion for Convergence	18
1	3.5 Sequences of the Arithmetic and Geometric Means	19
1	4 Indeterminate Forms and Techniques of Their Solution	22
1	4.1 Definition of Indeterminate Forms.....	22
1	4.2 Techniques of Solution of Indeterminate Forms	24
1	4.3 Various Indeterminate Forms and Examples	36
1	Exercises.....	38
2	Series of Numbers.....	43
2	1 Convergence and Introductory Examples	43
2	1.1 Definition of a Series. Partial Sums and Convergence	43
2	1.2 Elementary Examples of Series of Numbers	46
2	2 Elementary Properties of Convergent Series.....	52
2	2.1 Arithmetic Properties	53
2	2.2 Cauchy Criterion for Convergence	54
2	2.3 Necessary Condition of Convergence (Divergence Test)	55
2	2.4 Series and Its Remainder	55

3	Convergence of Positive Series	57
3.1	General Criterion for Convergence	58
3.2	Integral Test (Cauchy-Maclauren Test).....	58
3.3	The Comparison Tests.....	63
3.4	The Cauchy Condensation Test	69
3.5	D'Alembert's Tests (The Ratio Tests)	72
3.6	Cauchy's Tests (The Root Tests).....	76
3.7	Comparison Between D'Alembert's and Cauchy's Tests	79
3.8	Complement: Finer Forms of D'Alembert's and Cauchy's Tests	82
3.9	Complement: The Kummer Chain of Tests	85
3.10	Complement: The Cauchy Chain of Tests	97
4	Series of Different Types	104
4.1	Alternating Series.....	104
4.2	Dirichlet's and Abel's Tests	108
4.3	Absolute and Conditional Convergence	114
4.4	Product of Two Series	115
5	Associative and Commutative Properties of Series.....	118
5.1	Positive and Negative Parts of Series	118
5.2	Associative Property of Convergent Series.....	120
5.3	Commutative Property of Absolutely Convergent Series.....	122
5.4	Commutative Property of Conditionally Convergent Series.....	124
6	Complement: Double and Repeated Series	131
	Exercises	135
3	Sequences of Functions	141
1	Pointwise Convergence and Introductory Examples	141
2	Uniform and Non-uniform Convergence.....	149
2.1	Concept of the Uniform and Non-uniform Convergence	149
2.2	Arithmetic Properties of Uniform Convergence	155
2.3	Cauchy Criterion for Uniform Convergence	158
3	Dini's Theorem	160
4	Properties of Limit Functions Under Uniform Convergence.....	161
4.1	Boundedness of Limit Function	161
4.2	Limit of the Limit Function	163
4.3	Continuity of the Limit Function	166
4.4	Integrability of the Limit Function (Integration by Parameter)	169
4.5	Differentiability of the Limit Function (Differentiation by Parameter)	176
5	Complement: The Weierstrass Approximation Theorem	182
	Exercises	185

4	Series of Functions	191
1	Pointwise Convergence and Introductory Examples	191
2	Uniform and Non-uniform Convergence.....	197
2.1	Concept of Uniform and Non-uniform Convergence	197
2.2	Arithmetic Properties of Uniform Convergence	202
2.3	The Cauchy Criterion for Uniform Convergence	202
2.4	Uniform and Absolute Convergence.....	203
3	Sufficient Conditions for Uniform Convergence of Series	205
3.1	Comparison Tests	205
3.2	Dirichlet's and Abel's Tests	207
3.3	Dini's Theorem	214
4	Properties of the Sum of Uniformly Convergent Series	215
4.1	Boundedness of a Sum	215
4.2	Limit of a Sum	218
4.3	Continuity of a Sum	220
4.4	Integrability of a Sum (Integration Term by Term)	221
4.5	Differentiability of a Sum (Differentiation Term by Term).....	223
5	Complement: The Weierstrass Function—Everywhere Continuous and Nowhere Differentiable Function	227
	Exercises	231
5	Power Series	239
1	Introduction	239
2	Set of Convergence of a Power Series	242
2.1	Convergence of a Power Series	242
2.2	Determining the Radius of Convergence	245
2.3	Convergence of the Series of Derivatives	248
2.4	Behavior at the Endpoints of the Interval of Convergence	250
3	Properties of Power Series and Their Sums	251
3.1	Arithmetic Properties	251
3.2	Functional Properties	254
3.3	Analytic Properties	260
3.4	Uniqueness of Power Series Expansion, Analytic Functions	266
4	Taylor Series	269
4.1	Taylor Coefficients and Taylor Series	269
4.2	Relation Between the Taylor Series and Formula	271
4.3	Conditions of Expansion in the Taylor Series	274
5	Power Series Expansion of Elementary Functions	280
5.1	Using Analytic Properties of Power Series	281
5.2	Finding the Sum of Power Series via Differential Relations	284
5.3	Method of the Taylor Coefficients	287

5.4	Taylor Series for Various Functions	296
5.5	The List of the Derived Formulas of Taylor Series	304
6	Applications of Taylor Series	307
6.1	Approximation of Functions	308
6.2	Numerical Approximations	321
6.3	Finding Sums of Series of Functions	326
6.4	Sums of Series of Numbers	328
6.5	Calculation of Limits	329
6.6	Calculation of Integrals	336
6.7	Solution of Ordinary Differential Equations	339
6.8	Complement: The Number e Is Irrational	349
6.9	Complement: The Number π Is Irrational	350
7	Complement: Borel's Theorem	352
7.1	Smooth Non-analytic Function	353
7.2	Transition Function	355
7.3	Borel's Theorem	358
	Exercises	361
	Bibliography	369
	Index	373