CONTENTS

1 Geometry and Complex Arithmetic 1

1.1 Introduction 1

1.1.1 Historical Sketch 1

1.1.2 Bombelli's "Wild Thought" 4

1.1.3 Some Terminology and Notation 6

1.1.4 Practice 8

1.1.5 Equivalence of Symbolic and Geometric Arithmetic 8

1.2 Euler's Formula 11

1.2.1 Introduction 11

1.2.2 Moving Particle Argument 12

1.2.3 Power Series Argument 13

1.2.4 Sine and Cosine in Terms of Euler's Formula 15

1.3 Some Applications 16

1.3.1 Introduction 16

1.3.2 Trigonometry 16

1.3.3 Geometry 18

1.3.4 Calculus 22

1.3.5 Algebra 25

1.3.6 Vectorial Operations 30

1.4 Transformations and Euclidean Geometry* 33

1.4.1 Geometry Through the Eyes of Felix Klein 33

1.4.2 Classifying Motions 38

1.4.3 Three Reflections Theorem 41

1.4.4 Similarities and Complex Arithmetic 44

1.4.5 Spatial Complex Numbers? 48

1.5 Exercises 51

2 Complex Functions as Transformations 61

2.1 Introduction 61

2.2 Polynomials 64

2.2.1 Positive Integer Powers 64

2.2.2 Cubics Revisited* 66
2.2.3 Cassinian Curves* 67

2.3 Power Series 71
2.3.1 The Mystery of Real Power Series 71
2.3.2 The Disc of Convergence 75
2.3.3 Approximating a Power Series with a Polynomial 78
2.3.4 Uniqueness 79
2.3.5 Manipulating Power Series 81
2.3.6 Finding the Radius of Convergence 83
2.3.7 Fourier Series* 86

2.4 The Exponential Function 88
2.4.1 Power Series Approach 88
2.4.2 The Geometry of the Mapping 89
2.4.3 Another Approach 90

2.5 Cosine and Sine 94
2.5.1 Definitions and Identities 94
2.5.2 Relation to Hyperbolic Functions 95
2.5.3 The Geometry of the Mapping 97

2.6 Multifunctions 100
2.6.1 Example: Fractional Powers 100
2.6.2 Single-Valued Branches of a Multifunction 103
2.6.3 Relevance to Power Series 106
2.6.4 An Example with Two Branch Points 108

2.7 The Logarithm Function 110
2.7.1 Inverse of the Exponential Function 110
2.7.2 The Logarithmic Power Series 112
2.7.3 General Powers 113

2.8 Averaging over Circles* 115
2.8.1 The Centroid 115
2.8.2 Averaging over Regular Polygons 118
2.8.3 Averaging over Circles 121

2.9 Exercises 125

3 Möbius Transformations and Inversion 137

3.1 Introduction 137
3.1.1 Definition and Significance of Möbius Transformations 137

3.1.2 The Connection with Einstein's Theory of Relativity* 138
3.1.3 Decomposition into Simple Transformations 139

3.2 Inversion 139
3.2.1 Preliminary Definitions and Facts 139
3.2.2 Preservation of Circles 142
3.2.3 Constructing Inverse Points Using Orthogonal Circles 144
3.2.4 Preservation of Angles 147
3.2.5 Preservation of Symmetry 149
3.2.6 Inversion in a Sphere 150

3.3 Three Illustrative Applications of Inversion 153
3.3.1 A Problem on Touching Circles 153
3.3.2 A Curious Property of Quadrilaterals with Orthogonal Diagonals 154
3.3.3 Ptolemy's Theorem 156

3.4 The Riemann Sphere 157
3.4.1 The Point at Infinity 157
3.4.2 Stereographic Projection 158
3.4.3 Transferring Complex Functions to the Sphere 162
3.4.4 Behaviour of Functions at Infinity 163
3.4.5 Stereographic Formulae* 165

3.5 Möbius Transformations: Basic Results 168
3.5.1 Preservation of Circles, Angles, and Symmetry 168
3.5.2 Non-Uniqueness of the Coefficients 169
3.5.3 The Group Property 170
3.5.4 Fixed Points 171
3.5.5 Fixed Points at Infinity 172
3.5.6 The Cross-Ratio 174

3.6 Möbius Transformations as Matrices* 177
3.6.1 Empirical Evidence of a Link with Linear Algebra 177
3.6.2 The Explanation: Homogeneous Coordinates 178
3.6.3 Eigenvectors and Eigenvalues* 180
3.6.4 Rotations of the Sphere as Möbius Transformations* 182

3.7 Visualization and Classification* 184
3.7.1 The Main Idea 184

3.7.2 Elliptic, Hyperbolic, and Loxodromic Transformations 186
3.7.3 Local Geometric Interpretation of the Multiplier 189
3.7.4 Parabolic Transformations 190
3.7.5 Computing the Multiplier* 191
3.7.6 Eigenvalue Interpretation of the Multiplier* 192
3.8 Decomposition into 2 or 4 Reflections* 194
3.8.1 Introduction 194
3.8.2 Elliptic Case 194
3.8.3 Hyperbolic Case 196
3.8.4 Parabolic Case 197
3.8.5 Summary 198
3.9 Automorphisms of the Unit Disc* 199
3.9.1 Counting Degrees of Freedom 199
3.9.2 Finding the Formula via the Symmetry Principle 200
3.9.3 Interpreting the Simplest Formula Geometrically* 201
3.9.4 Introduction to Riemann's Mapping Theorem 204
3.10 Exercises 205

4 Differentiation: The Amplitwist Concept 213
4.1 Introduction 213
4.2 A Puzzling Phenomenon 213
4.3 Local Description of Mappings in the Plane 216
4.3.1 Introduction 216
4.3.2 The Jacobian Matrix 217
4.3.3 The Amplitwist Concept 218
4.4 The Complex Derivative as Amplitwist 220
4.4.1 The Real Derivative Re-examined 220
4.4.2 The Complex Derivative 221
4.4.3 Analytic Functions 223
4.4.4 A Brief Summary 224
4.5 Some Simple Examples 225
4.6 Conformal = Analytic 227
4.6.1 Introduction 227
4.6.2 Conformality Throughout a Region 228
4.6.3 Conformality and the Riemann Sphere 230
4.7 Critical Points 231
4.7.1 Degrees of Crushing 231

4.7.2 Breakdown of Conformality 232
4.7.3 Branch Points 233
4.8 The Cauchy–Riemann Equations 234
4.8.1 Introduction 234
4.8.2 The Geometry of Linear Transformations 235
4.8.3 The Cauchy–Riemann Equations 237
4.9 Exercises 239

5 Further Geometry of Differentiation 245
5.1 Cauchy–Riemann Revealed 245
5.1.1 Introduction 245
5.1.2 The Cartesian Form 245
5.1.3 The Polar Form 247
5.2 An Intimation of Rigidity 249
5.3 Visual Differentiation of log(z) 252
5.4 Rules of Differentiation 254
5.4.1 Composition 254
5.4.2 Inverse Functions 255
5.4.3 Addition and Multiplication 256
5.5 Polynomials, Power Series, and Rational Functions 257
5.5.1 Polynomials 257
5.5.2 Power Series 257
5.5.3 Rational Functions 259
5.6 Visual Differentiation of the Power Function 260
5.7 Visual Differentiation of exp(z) 262
5.8 Geometric Solution of $E' = E$ 264
5.9 An Application of Higher Derivatives: Curvature* 266
5.9.1 Introduction 266
5.9.2 Analytic Transformation of Curvature 267
5.9.3 Complex Curvature 270
5.10 Celestial Mechanics* 274
5.10.1 Central Force Fields 274
5.10.2 Two Kinds of Elliptical Orbit 274
5.10.3 Changing the First into the Second 277
5.10.4 The Geometry of Force 278
5.10.5 An Explanation 279
5.10.6 The Kasner–Arnol'd Theorem 280

5.11 Analytic Continuation* 281
5.11.1 Introduction 281
5.11.2 Rigidity 283
5.11.3 Uniqueness 284
5.11.4 Preservation of Identities 286
5.11.5 Analytic Continuation via Reflections 286
5.12 Exercises 293

6 Non-Euclidean Geometry* 303
6.1 Introduction 303
6.1.1 The Parallel Axiom 303
6.1.2 Some Facts from Non-Euclidean Geometry 305
6.1.3 Geometry on a Curved Surface 307
6.1.4 Intrinsic versus Extrinsic Geometry 309
6.1.5 Gaussian Curvature 311
6.1.6 Surfaces of Constant Curvature 313
6.1.7 The Connection with Möbius Transformations 315
6.2 Spherical Geometry 316
6.2.1 The Angular Excess of a Spherical Triangle 316
6.2.2 Motions of the Sphere: Spatial Rotations and Reflections 317
6.2.3 A Conformal Map of the Sphere 321
6.2.4 Spatial Rotations as Möbius Transformations 325
6.2.5 Spatial Rotations and Quaternions 329
6.3 Hyperbolic Geometry 333
6.3.1 The Tractrix and the Pseudosphere 333
6.3.2 The Constant Negative Curvature of the Pseudosphere* 335
6.3.3 A Conformal Map of the Pseudosphere 336
6.3.4 Beltrami's Hyperbolic Plane 339
6.3.5 Hyperbolic Lines and Reflections 342
6.3.6 The Bolyai–Lobachevsky Formula* 347
6.3.7 The Three Types of Direct Motion 348
6.3.8 Decomposing an Arbitrary Direct Motion into Two Reflections 353
6.3.9 The Angular Excess of a Hyperbolic Triangle 357
6.3.10 The Beltrami–Poincaré Disc 359
6.3.11 Motions of the Beltrami–Poincaré Disc 363
6.3.12 The Hemisphere Model and Hyperbolic Space 367
6.4 Exercises 374

7 Winding Numbers and Topology 385
7.1 Winding Number 385
7.1.1 The Definition 385
7.1.2 What Does "Inside" Mean? 386
7.1.3 Finding Winding Numbers Quickly 387
7.2 Hopf's Degree Theorem 388
7.2.1 The Result 388
7.2.2 Loops as Mappings of the Circle* 390
7.2.3 The Explanation* 391
7.3 Polynomials and the Argument Principle 393
7.4 A Topological Argument Principle* 394
7.4.1 Counting Preimages Algebraically 394
7.4.2 Counting Preimages Geometrically 396
7.4.3 What's Topologically Special About Analytic Functions? 398
7.4.4 A Topological Argument Principle 399
7.4.5 Two Examples 401
7.5 Rouché's Theorem 403
7.5.1 The Result 403
7.5.2 The Fundamental Theorem of Algebra 404
7.5.3 Brouwer's Fixed Point Theorem* 404
7.6 Maxima and Minima 405
7.6.1 Maximum-Modulus Theorem 405
7.6.2 Related Results 407
7.7 The Schwarz–Pick Lemma* 407
7.7.1 Schwarz's Lemma 407
7.7.2 Liouville's Theorem 409
7.7.3 Pick's Result 411
7.8 The Generalized Argument Principle 414
7.8.1 Rational Functions 414
7.8.2 Poles and Essential Singularities 416
7.8.3 The Explanation* 419
7.9 Exercises 420

8 Complex Integration: Cauchy's Theorem 429
8.1 Introduction 429
8.2 The Real Integral 430
8.2.1 The Riemann Sum 430

8.2.2 The Trapezoidal Rule 432
8.2.3 Geometric Estimation of Errors 434

8.3 The Complex Integral 436
8.3.1 Complex Riemann Sums 436
8.3.2 A Visual Technique 439
8.3.3 A Useful Inequality 440
8.3.4 Rules of Integration 441

8.4 Complex Inversion 442
8.4.1 A Circular Arc 442
8.4.2 General Loops 444
8.4.3 Winding Number 446

8.5 Conjugation 447
8.5.1 Introduction 447
8.5.2 Area Interpretation 448
8.5.3 General Loops 450

8.6 Power Functions 450
8.6.1 Integration along a Circular Arc 450
8.6.2 Complex Inversion as a Limiting Case* 452
8.6.3 General Contours and the Deformation Theorem 453
8.6.4 A Further Extension of the Theorem 454
8.6.5 Residues 455

8.7 The Exponential Mapping 457

8.8 The Fundamental Theorem 458
8.8.1 Introduction 458
8.8.2 An Example 459
8.8.3 The Fundamental Theorem 460
8.8.4 The Integral as Antiderivative 462
8.8.5 Logarithm as Integral 465

8.9 Parametric Evaluation 466

8.10 Cauchy's Theorem 467
8.10.1 Some Preliminaries 467
8.10.2 The Explanation 469

8.11 The General Cauchy Theorem 472
8.11.1 The Result 472
8.11.2 The Explanation 473
8.11.3 A Simpler Explanation 474

8.12 The General Formula of Contour Integration 475

8.13 Exercises 478

9 Cauchy's Formula and Its Applications 485

9.1 Cauchy's Formula 485
9.1.1 Introduction 485
9.1.2 First Explanation 486
9.1.3 Gauss's Mean Value Theorem 487
9.1.4 A Second Explanation and the General Cauchy Formula 488

9.2 Infinite Differentiability and Taylor Series 489
9.2.1 Infinite Differentiability 489
9.2.2 Taylor Series 491

9.3 Calculus of Residues 493
9.3.1 Laurent Series Centred at a Pole 493
9.3.2 A Formula for Calculating Residues 494
9.3.3 Application to Real Integrals 495
9.3.4 Calculating Residues using Taylor Series 497
9.3.5 Application to Summation of Series 498

9.4 Annular Laurent Series 501
9.4.1 An Example 501
9.4.2 Laurent's Theorem 502

9.5 Exercises 506

10 Vector Fields: Physics and Topology 511

10.1 Vector Fields 511
10.1.1 Complex Functions as Vector Fields 511
10.1.2 Physical Vector Fields 513
10.1.3 Flows and Force Fields 515
10.1.4 Sources and Sinks 516

10.2 Winding Numbers and Vector Fields* 518
10.2.1 The Index of a Singular Point 518
10.2.2 The Index According to Poincaré 522
10.2.3 The Index Theorem 523

10.3 Flows on Closed Surfaces* 525
10.3.1 Formulation of the Poincaré–Hopf Theorem 525
10.3.2 Defining the Index on a Surface 527
10.3.3 An Explanation of the Poincaré–Hopf Theorem 529

10.4 Exercises 532

11 Vector Fields and Complex Integration 537
11.1 Flux and Work 537
11.1.1 Flux 537
11.1.2 Work 539
11.1.3 Local Flux and Local Work 542
11.1.4 Divergence and Curl in Geometric Form* 543
11.1.5 Divergence-Free and Curl-Free Vector Fields 545
11.2 Complex Integration in Terms of Vector Fields 547
11.2.1 The Pólya Vector Field 547
11.2.2 Cauchy's Theorem 549
11.2.3 Example: Area as Flux 550
11.2.4 Example: Winding Number as Flux 551
11.2.5 Local Behaviour of Vector Fields* 553
11.2.6 Cauchy's Formula 555
11.2.7 Positive Powers 556
11.2.8 Negative Powers and Multipoles 557
11.2.9 Multipoles at Infinity 560
11.2.10 Laurent's Series as a Multipole Expansion 561
11.3 The Complex Potential 562
11.3.1 Introduction 562
11.3.2 The Stream Function 563
11.3.3 The Gradient Field 565
11.3.4 The Potential Function 567
11.3.5 The Complex Potential 569
11.3.6 Examples 572
11.4 Exercises 574

12 Flows and Harmonic Functions 577
12.1 Harmonic Duals 577
12.1.1 Dual Flows 577
12.1.2 Harmonic Duals 580
12.2 Conformal Invariance 583
12.2.1 Conformal Invariance of Harmonicity 583
12.2.2 Conformal Invariance of the Laplacian 584
12.2.3 The Meaning of the Laplacian 586
12.3 A Powerful Computational Tool 587
12.4 The Complex Curvature Revisited* 590
12.4.1 Some Geometry of Harmonic Equipotentials 590
12.4.2 The Curvature of Harmonic Equipotentials 590

12.4.3 Further Complex Curvature Calculations 594
12.4.4 Further Geometry of the Complex Curvature 595
12.5 Flow Around an Obstacle 598
12.5.1 Introduction 598
12.5.2 An Example 599
12.5.3 The Method of Images 604
12.5.4 Mapping One Flow Onto Another 611
12.6 The Physics of Riemann's Mapping Theorem 613
12.6.1 Introduction 613
12.6.2 Exterior Mappings and Flows Round Obstacles 615
12.6.3 Interior Mappings and Dipoles 618
12.6.4 Interior Mappings, Vortices, and Sources 620
12.6.5 An Example: Automorphisms of the Disc 624
12.6.6 Green's Function 626
12.7 Dirichlet's Problem 630
12.7.1 Introduction 630
12.7.2 Schwarz's Interpretation 631
12.7.3 Dirichlet's Problem for the Disc 634
12.7.4 The Interpretations of Neumann and Bôcher 637
12.7.5 Green's General Formula 643
12.8 Exercises 649

Bibliography 653
Index 661